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UMBILICAL SUBMANIFOLDS WITH RESPECT TO A NONPARALLEL NORMAL DIRECTION

BANG-YEN CHEN & KENTARO YANO

Let M^n be an n-dimensional submanifolds[1] of an $(n+2)$-dimensional euclidean space E^{n+2}, and C be a unit normal vector field of M^n in E^{n+2}. If the second fundamental tensor in the normal direction C is proportional to the first fundamental tensor of the submanifold M^n, then M^n is said to be *umbilical* with respect to the normal direction C. The normal direction C is said to be *parallel* if the covariant differentiation of C along M^n has no normal component, and C is said to be *nonparallel* if the covariant differentiation of C along M^n has nonzero normal component everywhere.

In a previous paper [1], the authors proved that a submanifold is umbilical with respect to a parallel normal direction C if and only if it is contained either in a hypersphere or in a hyperplane of the euclidean space. In the present paper, we shall study the submanifolds of codimension 2 of a euclidean space which are umbilical with respect to a nonparallel normal direction.

1. Preliminaries

We consider a submanifold M^n of codimension 2 of an $(n+2)$-dimensional euclidean space E^{n+2}, and represent it by

$$X = X(\xi^1, \cdots, \xi^n) , \tag{1}$$

where X is the position vector from the origin of E^{n+2} to a point of the submanifold M^n, and $\{\xi^h\}$ is a local coordinate system in M^n, where and throughout this paper the indices $h, i, j, k, \cdots$ run over the range $\{1, \cdots, n\}$.

Put

$$X_i = \partial_i X , \qquad \partial_i = \partial/\partial\xi^i , \tag{2}$$

and denote by C and D two mutually orthogonal unit normals to M^n. Then, denoting by ∇_j the operator of covariant differentiation with respect to the Riemannian metric $g_{ji} = X_j \cdot X_i$ of M^n, we have the equations of Gauss

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[1] Manifolds, mappings, functions, ... are assumed to be sufficiently differentiable, and we shall restrict discussions only to manifolds of dimension $n>2$.

$$\nabla_j X_i \equiv \partial_j X_i - \left\{ {h \atop ji} \right\} X_h = h_{ji} C + k_{ji} D \ , \tag{3}$$

where $\left\{ {h \atop ji} \right\}$ are Christoffel symbols formed with g_{ji}, and h_{ji} and k_{ji} the second fundamental tensors with respect to the normals C and D respectively. The mean curvature vector is thus given by

$$H = n^{-1} g^{ji} \nabla_j X_i \ , \tag{4}$$

where g^{ji} are contravariant components of the metric tensor.

If there exist two functions α, β and a unit vector field u_i on the submanifold M^n such that

$$h_{ji} = \alpha g_{ji} + \beta u_j u_i \ , \tag{5}$$

then M^n is said to be *quasi-umbilical* with respect to the normal direction C. In particular, if $\beta = 0$ identically, then M^n is umbilical with respect to the normal direction C. If M^n is umbilical with respect to the mean curvature vector H, then M^n is said to be *pseudo-umbilical*.

The equations of Weingarten are given by

$$\nabla_j C = -h_j{}^i X_i + l_j D \ , \tag{6}$$

$$\nabla_j D = -k_j{}^i X_i - l_j C \ , \tag{7}$$

where $h_j{}^i = h_{jt} g^{ti}$, $k_j{}^i = k_{jt} g^{ti}$ and l_j the third fundamental tensor. The normal vector fields C and D are said to be *parallel* or *nonparallel* according as the third fundamental tensor vanishes or never vanishes.

We also have the equations of Gauss, Codazzi and Ricci respectively:

$$K_{kji}{}^h = h_k{}^h h_{ji} - h_j{}^h h_{ki} + k_k{}^h k_{ji} - k_j{}^h k_{ki} \ ; \tag{8}$$

$$\nabla_k h_{ji} - \nabla_j h_{ki} - l_k k_{ji} + l_j k_{ki} = 0 \ , \tag{9}$$

$$\nabla_k k_{ji} - \nabla_j k_{ki} + l_k h_{ji} - l_j h_{ki} = 0 \ ; \tag{10}$$

$$\nabla_j l_i - \nabla_i l_j + h_{jt} k_i{}^t - h_{it} k_j{}^t = 0 \ , \tag{11}$$

where $K_{kji}{}^h$ is the Riemann-Christofel curvature tensor.

Denoting the Ricci tensor and the scalar curvature respectively by $K_{ji} = K_{tji}{}^t$ and $K = g^{ji} K_{ji}$, we define a tensor L_{ji} of type (0, 2) by

$$L_{ji} = -\frac{K_{ji}}{n-2} + \frac{K g_{ji}}{2(n-1)(n-2)} \ . \tag{12}$$

The conformal curvature tensor $C_{kji}{}^h$ is then given by

$$C_{kji}{}^h = K_{kji}{}^h + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k{}^h g_{ji} - L_j{}^h g_{ki} , \tag{13}$$

where δ_k^h are Kronecker deltas, and $L_k{}^h = L_{kt} g^{th}$.

A Riemannian manifold M^n is called a *conformally flat space* if we have

$$C_{kji}{}^h = 0 , \tag{14}$$

$$\nabla_k L_{ji} - \nabla_j L_{ki} = 0 . \tag{15}$$

It is well-known that (14) holds automatically for $n = 3$, and (15) is a consequence of (14) for $n > 3$.

2. Submanifolds umbilical with respect to a normal direction

In the sequel, we always assume that C and D are two mutually orthogonal unit normals to M^n in E^{n+2}.

Theorem 1. *If a submanifold M^n of codimension 2 of a euclidean space is umbilical with respect to a nonparallel normal direction C, then M^n is quasi-umbilical with respect to another normal direction D.*

Proof. We assume that M^n is umbilical with respect to a normal direction C, and C is nonparallel. Then we have

$$h_{ji} = \alpha g_{ji} , \qquad l_j \neq 0 , \tag{16}$$

α being a function. Then from (9) and (16) it follows that

$$\alpha_k g_{ji} - \alpha_j g_{ki} - l_k k_{ji} + l_j k_{ki} = 0 , \tag{17}$$

where $\alpha_k = \partial_k \alpha$. Transvecting l^i to (17) and l^k to the resulting equation, we obtain

$$\alpha_j + k_{jt} l^t = l^{-2}(\alpha_t l^t + k(l, l)) l_j , \tag{18}$$

where

$$k(l, l) = k_{ts} l^t l^s , \qquad l^2 = l_i l^i .$$

Transvecting g^{ki} to (17) gives

$$\alpha_j + k_{jt} l^t = -(n - 2)\alpha_j + k_t{}^t l_j , \tag{19}$$

from which by transvecting l^j we obtain

$$(n - 1)\alpha_t l^t + k(l, l) = k_t{}^t l^2 . \tag{20}$$

By eliminating $\alpha_j + k_{jt} l^t$ from (18) and (19), and using (20) we easily find

$$\alpha_j = l^{-2}(\alpha_t l^t) l_j . \tag{21}$$

Substitution of (21) into (19) and use of (20) yield immediately

$$k_{jt}l^t = l^{-2}k(l, l)l_j \ . \tag{22}$$

Transvecting l^k to (17), and substituting (21) and (22) into the resulting equation, we have

$$k_{ji} = \lambda g_{ji} + \mu l_j l_i \ , \tag{23}$$

where

$$\lambda = \alpha_t l^t / l^2 \ , \qquad \mu = (k(l, l) - \alpha_t l^t)/l^4 = (k_t{}^t - n\lambda)/l^2 \tag{24}$$

by (20). This proves the theorem.

Proposition 2. *Under the hypothesis of Theorem* 1, *we have*

$$\alpha_i = \lambda l_i \ . \tag{25}$$

This proposition follows immediately from (21) and the definition (24) of λ.

3. Conformally flat spaces of codimension 2

The purpose of this section is to prove

Theorem 3. *If a submanifold of codimension* 2 *of a euclidean* $(n + 2)$-*space is umbilical with respect to a nonparallel normal direction* C, *then it is conformally flat.*

Proof. Since the submanifold is umbilical with respect to the normal direction C and C is nonparallel, we have

$$h_{ji} = \alpha g_{ji} \ , \qquad l_j \neq 0 \ .$$

We consider the cases $n > 3$ and $n = 3$ separately.

Case I: $n > 3$. By substituting (16) and (23) into (8), we find

$$\begin{aligned} K_{kji}{}^h = {} & (\alpha^2 + \lambda^2)(\delta_k^h g_{ji} - \delta_j^h g_{ki}) \\ & + \lambda\mu[(\delta_k^h l_j - \delta_j^h l_k)l_i + (l_k g_{ji} - l_j g_{ki})l^h] \ , \end{aligned} \tag{26}$$

from which follow

$$K_{ji} = [(n - 1)(\alpha^2 + \lambda^2) + \lambda\mu l^2]g_{ji} + (n - 2)\lambda\mu l_j l_i \ , \tag{27}$$

$$K = n(n - 1)(\alpha^2 + \lambda^2) + 2(n - 1)\lambda\mu l^2 \ . \tag{28}$$

Thus from (12), (27) and (28) we have

$$L_{ji} = -\tfrac{1}{2}(\alpha^2 + \lambda^2)g_{ji} - \lambda\mu l_j l_i \ . \tag{29}$$

Substituting (26) and (29) into (13), we easily find that the conformal curvature tensor $C_{kji}{}^{h}$ vanishes identically. This shows that the submanifold M^n is a conformally flat space for $n > 3$.

Case II: $n = 3$. Substituting (16) and (23) into (10), and using (11) we obtain

$$\begin{aligned} \lambda_k g_{ji} - \lambda_j g_{ki} + \mu_k l_j l_i - \mu_j l_k l_i + \mu l_i \nabla_k l_i \\ - \mu l_k \nabla_j l_i + l_k \alpha g_{ji} - l_j \alpha g_{ki} = 0 \,, \end{aligned} \tag{30}$$

where $\lambda_k = \partial_k \lambda$ and $\mu_k = \partial_k \mu$.

Transvecting l^k to (30) gives

$$\begin{aligned} \lambda_t l^t g_{ji} - \lambda_j l_i + \mu_t l^t l_j l_i - \mu_j l^2 l_i + \mu l_j l^k \nabla_k l_i \\ - \mu l^2 \nabla_j l_i + l^2 \alpha g_{ji} - \alpha l_j l_i = 0 \,, \end{aligned} \tag{31}$$

which shows that $\mu \nabla_j l_i$ is of the form

$$\mu \nabla_j l_i = p g_{ji} + q_j l_i + q_i l_j \,, \tag{32}$$

where

$$p = \lambda_t l^t / l^2 + \alpha \,, \tag{33}$$

since $\mu \nabla_j l_i$ is symmetric by (11).

Substituting (32) into (30) we find

$$\begin{aligned} [\lambda_k + (\alpha - p) l_k] g_{ji} - [\lambda_j + (\alpha - p) l_j] g_{ki} \\ + (\mu_k l_j - \mu_j l_k + q_k l_j - q_j l_k) l_i = 0 \,, \end{aligned}$$

from which follow

$$\lambda_k + (\alpha - p) l_k = 0 \,, \tag{34}$$

$$(\mu_k + q_k) l_j - (\mu_j + q_j) l_k = 0 \,. \tag{35}$$

From (33) and (34) we find

$$\lambda_k = l^{-2} (\lambda_t l^t) l_k \,. \tag{36}$$

(35) implies

$$\mu_j + q_j = r l_j \,, \tag{37}$$

r being a function. Substituting (33) and (37) into (32) gives

$$\mu \nabla_j l_i = (\lambda_t l^t / l^2 + \alpha) g_{ji} - (\mu_j l_i + \mu_i l_j) + 2 r l_j l_i \,. \tag{38}$$

Thus from (25), (29), (36), (38), by a straightforward computation we find

$$\nabla_k L_{ji} - \nabla_j L_{ki} = 0 ,$$

which shows that M^n is a conformally flat space. Consequently we have completely proved the theorem.

4. Locus of $(n-1)$-spheres

The purpose of this section is to prove

Theorem 4. *If a submanifold of codimension 2 of a euclidean space is umbilical with respect to a nonparallel normal direction C, then it is the locus of $(n-1)$-spheres, where an $(n-1)$-sphere means a hypersphere or a hyperplane of a euclidean n-space.*

Proof. Let the submanifold M^n be umbilical with respect to the normal direction C, and C be nonparallel. Then the formulas in § 2 and § 3 are all valid. Since $\nabla_j l_i - \nabla_i l_j = 0$, the distribution $l_i dx^i = 0$ is integrable. We represent one of the integral manifolds M^{n-1} of this distribution by $\xi^h = \xi^h(\eta^a)$, and put

$$B_b{}^h = \partial_b \xi^h , \quad N^h = l^h / l , \quad \partial_b = \partial / \partial \eta^b ,$$
$$g_{cb} = B_c{}^j B_b{}^i g_{ji} , \qquad \nabla_c B_b{}^h = H_{cb} N^h ,$$

$\nabla_c B_b{}^h$ denoting the van der Waerden-Bortolotti covariant differentiation of $B_b{}^h$ along M^{n-1}:

$$\nabla_c B_b{}^h = \partial_c B_b{}^h + B_c{}^j B_b{}^i \left\{ {h \atop ji} \right\} - B_a{}^h \left\{ {a \atop cb} \right\} ,$$

where $\left\{ {a \atop cb} \right\}$ are Christoffel symbols formed with g_{cb}, and H_{cb} is the second fundamental tensor of M^{n-1}. Here and in the sequel, the indices $a, b, c, \cdots$ run over the range $\{1, \cdots, n-1\}$. From Proposition 2 and (36) it follows that along M^{n-1}

(39) $$\alpha = \text{const.}$$

(40) $$\lambda = \text{const.}$$

respectively. Now putting

(41) $$X_b = \partial_b X = B_b{}^i X_i ,$$

we have, in consequence of (3),

(42) $$\begin{aligned} \nabla_c X_b &= H_{cb} N^i X_i + B_c{}^j B_b{}^i (h_{ji} C + k_{ji} D) \\ &= \alpha g_{cb} C + \lambda g_{cb} D + H_{cb} N , \end{aligned}$$

where $N = N^i X_i$.

From (6) it follows that

$$\nabla_c C = B_c{}^j \nabla_j C = B_c{}^j(-\alpha X_j + l_j D) ,$$

that is,

$$\nabla_c C = -\alpha X_c . \tag{43}$$

Similarly, from (7) and (23) we have

$$\nabla_c D = B_c{}^j \nabla_j D = B_c{}^j(-\lambda X_j + \mu l_j l^i X_i + l_j C) ,$$

that is,

$$\nabla_c D = -\lambda X_c . \tag{44}$$

We also have

$$\begin{aligned} \nabla_c N = \nabla_c(N^i X_i) &= (-H_c{}^a B_a{}^i) X_i + B_c{}^j N^i (\nabla_j X_i) \\ &= -H_c{}^a X_a + B_c{}^j N^i[\alpha g_{ji} C + (\lambda g_{ji} + \mu l_j l_i) D] , \end{aligned}$$

that is,

$$\nabla_c N = -H_c{}^a X_a . \tag{45}$$

From (38) it follows that

$$B_c{}^j B_b{}^i (\mu \nabla_j l_i) = (\lambda_t l^t / l^2 + \alpha) B_c{}^j B_b{}^i g_{ji} ,$$

which implies

$$\mu[\nabla_c(l_i B_b{}^i) - l_i \nabla_c B_b{}^i] = (\lambda_t l^t / l^2 + \alpha) g_{cb} ,$$

that is,

$$\mu l H_{cb} = -(\lambda_t l^t / l^2 + \alpha) g_{cb} .$$

Let U denote the open subset of M^n in which $\mu \neq 0$, and V the interior of $M^n - U$. Then from (16) and (23) we see that V is totally umbilical in the euclidean $(n + 2)$-space E^{n+1}, so that every component of V is contained either in a hypersphere of E^{n+2} or in a hyperplane of E^{n+2}. Thus the closure of $V = M - U$ is a locus of $(n - 1)$-spheres. Since on the subset U we have $H_{cb} = \nu g_{cb}$, v being a function, (45) becomes

$$\nabla_c N = -\nu X_c , \tag{46}$$

from which follows

(47) $$\nu = \text{const}$$

so that

(48) $$\nabla_c X_b = \alpha g_{cb} C + \lambda g_{cb} D + \nu g_{cb} N ,$$

α, λ, ν being constants. Thus if $\mu \neq 0$, then M^{n-1} is an $(n-1)$-sphere. This implies that U is also the locus of $(n-1)$-spheres. Hence the proof of the theorem is complete.

5. $h_{ji} = \alpha g_{ji}$ with α = constant

In this section we shall study submanifolds of codimension 2 of a euclidean space, which are umbilical with respect to a nonparallel normal direction C with $h_{ji} = \alpha g_{ji}$ and α = constant. The main results are the following two theorems.

Theorem 5. *If a submanifold of codimension 2 of a euclidean space is umbilical with respect to a nonparallel normal direction C with $h_{ji} = \alpha g_{ji}$ and α = constant, then the submanifold is of constant curvature α^2.*

Proof. Suppose that M^n is umbilical with respect to a normal direction C, $h_{ji} = \alpha$, α = constant and C is nonparallel. Then

(49) $$\alpha_j = 0 , \qquad l_j \neq 0 ,$$

which reduces the first equation of (24) to

(50) $$\lambda = 0 .$$

Substitution of (50) into (23) gives

(51) $$h_{ji} = \alpha g_{ji} , \qquad k_{ji} = \mu l_j l_i .$$

Thus from (8) and (51) we obtain

$$K_{kji}{}^h = \alpha^2(\delta_k^h g_{ji} - \delta_j^h g_{ki}) ,$$

which proves the theorem.

Theorem 6. *If a submanifold of codimension 2 of a euclidean space is geodesic with respect to a nonparallel normal direction C, then the submanifold is the locus of $(n-1)$-planes. In particular, if the submanifold is complete, then it is a cylinder.*

Proof. If the submanifold M^n is geodesic with respect to the normal direction C, and C is nonparallel, then

(52) $$h_{ji} = 0 , \qquad l_j \neq 0 ,$$

so that

(53) $$\alpha = 0\,, \qquad \lambda = 0\,,$$

which reduces (30) to

(54) $$\mu_k l_j l_i - \mu_j l_k l_i + \mu l_j \nabla_k l_i - \mu l_k \nabla_j l_i = 0\,.$$

As we see in the proof of Theorem 4, the distribution $l_i dx^i = 0$ is completely integrable. If we represent one of the integral manifolds M^{n-1} of this distribution by $\xi^h = \xi^h(\eta^a)$, and put

$$B_b{}^h = \partial_b \xi^h\,, \quad N^h = l^h/l\,, \quad \nabla_c B_b{}^h = H_{cb} N^h\,,$$

then transvecting $B_a{}^k N^j B_b{}^i$ to (54) we find

$$\mu l_j N^j B_a{}^k B_b{}^i (\nabla_k l_i) = 0\,,$$

that is,

(55) $$\mu l^2 H_{ab} = 0\,.$$

Let U denote the open subset of M^n in which $\mu \neq 0$, and V the interior of $M^n - U$. Then we see from (16), (23) and (50) that V is totally geodesic in E^{n+2}, so that every component of V is contained in a euclidean n-space in E^{n+2}. Thus V is the locus of euclidean $(n-1)$-spaces. Since $H_{ab} = 0$ on the subset U, we have $\nabla_c X_b = 0$, which implies that M^{n-1} is contained in a euclidean $(n-1)$-space. Consequently the submanifold M^n is the locus of euclidean $(n-1)$-spaces.

If the submanifold is complete, then by the flatness of the submanifold we see that M^n is a cylinder. This completes the proof of the theorem.

Bibliography

[1] B. Y. Chen & K. Yano, *Integral formulas for submanifolds and their applications,* J. Differential Geometry 5 (1971) 467–477.

MICHIGAN STATE UNIVERSITY